Sample Solution

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Question 4. Convert the pair of second-order equations

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 3\frac{\mathrm{d}z}{\mathrm{d}t} + 2y = 0, \qquad \frac{\mathrm{d}^2 z}{\mathrm{d}t^2} + 3\frac{\mathrm{d}y}{\mathrm{d}t} + 2z = 0 \tag{1}$$

into a system of 4 first-order equations for the variables

$$x_1 = y, \qquad x_2 = y', \qquad x_3 = z, \qquad x_4 = z'. \tag{2}$$

Solution. From (2) we know that

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= \frac{\mathrm{d}y}{\mathrm{d}t} = y' = x_2, \\ \frac{\mathrm{d}x_3}{\mathrm{d}t} &= \frac{\mathrm{d}z}{\mathrm{d}t} = z' = x_4. \end{aligned}$$

Since

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = y'' = \frac{\mathrm{d}x_2}{\mathrm{d}t}, \qquad \frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = z'' = \frac{\mathrm{d}x_4}{\mathrm{d}t}$$

plug into (1) we know that

$$\begin{aligned} \frac{\mathrm{d}x_2}{\mathrm{d}t} + 3x_4 + 2x_1 = 0, \\ \frac{\mathrm{d}x_4}{\mathrm{d}t} + 3x_2 + 2x_3 = 0. \end{aligned}$$

Therefore

$$\boxed{\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -2x_1 - 3x_4, \\ \dot{x}_3 = x_4, \\ \dot{x}_4 = -3x_2 - 2x_3. \end{cases}}$$

Question 6. Write the given system of differential equations and initial values in the form of $\dot{\boldsymbol{x}} = A\boldsymbol{x}$, $\boldsymbol{x}(t_0) = \boldsymbol{x}^0$.

$$\dot{x}_1 = 3x_1 - 7x_2, \qquad x_1(0) = 1 \tag{3}$$
$$\dot{x}_2 = 4x_1, \qquad x_2(0) = 1 \tag{4}$$

Solution. Define

$$\boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

Then

$$\dot{\boldsymbol{x}} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 3x_1 - 7x_2 \\ 4x_1 \end{pmatrix} = \begin{pmatrix} 3 & -7 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 & -7 \\ 4 & 0 \end{pmatrix} \boldsymbol{x}.$$

At $t = 0$,

$$\boldsymbol{x}(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Therefore

$$\boxed{\dot{\boldsymbol{x}} = \begin{pmatrix} 3 & -7 \\ 4 & 0 \end{pmatrix} \boldsymbol{x}, \qquad \boldsymbol{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.}$$