Partial regularity results for the three-dimensional incompressible Navier–Stokes equation

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Literature review on the Navier–Stokes equation

3D incompressible Navier–Stokes equation

- Velocity $u(t,x) : [0,T) \times \mathbb{R}^3 \to \mathbb{R}^3$.
- Pressure $P(t,x) : [0,T) \times \mathbb{R}^3 \to \mathbb{R}$.

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = \Delta u \\ \operatorname{div} u = 0 \\ u\big|_{t=0} = u_0 \end{cases} \tag{NSE}$$

- Weak solution: $u \in \mathscr{D}'$, s.t. $\forall \varphi \in C^\infty([0,T) \times \mathbb{R}^3)$, $\operatorname{div} \varphi = 0$, $\operatorname{supp} \varphi \subset\subset \mathbb{R}^3 \times [0,T)$,

$$\int_0^T \int_{\mathbb{R}^3} -\partial_t \varphi \cdot u - (u \cdot \nabla \varphi) \cdot u + \nabla \varphi \cdot \nabla u \, \mathrm{d}x \, \mathrm{d}t = \int_{\mathbb{R}^3} u_0 \cdot \varphi\big|_{t=0} \, \mathrm{d}x.$$

3D Incompressible Navier–Stokes equation

- Leray–Hopf solution: a weak solution

$$u \in L^2(0,T;\dot H^1(\mathbb{R}^3)) \cap C_{\mathrm{w}}([0,T];L^2(\mathbb{R}^3)),$$

with energy inequality $\forall \tau \in (0,T)$,

$$\frac{1}{2}\int_{\mathbb{R}^3} |u(\tau,x)|^2 \,\mathrm{d}x + \int_0^\tau \int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}t \le \frac{1}{2}\int_{\mathbb{R}^3} |u_0|^2 \,\mathrm{d}x.$$

- Suitable weak solution: a Leray–Hopf solution with generalized energy inequality in the sense of distribution, $\forall t \in (0,T)$ a.e.,

$$\partial_t \frac{|u|^2}{2} + \operatorname{div}\left[u\left(\frac{|u|^2}{2} + P\right)\right] + |\nabla u|^2 - \Delta \frac{|u|^2}{2} \le 0.$$

Known results for suitable weak solutions

- Global-in-time existence (Scheffer, 1978
  Caffarelli—Kohn—Nirenberg, 1982)
- Partial regularity: $\mathscr{H}^1(\mathrm{Sing}(u)) = 0$
  (C—K—N, 1982; Lin, 1998; Vasseur, 2007
  Kukavica, 2008, 2011
  Chamorro—Lemarié-Rieusset—Mayoufi, 2018)
- Second derivative estimate: $\nabla^2 u \in L_{t,x}^{\frac{4}{3}-\varepsilon}$ (Constantin, 1990)
  $\nabla^2 u \in L_{t,x}^{\frac{4}{3},\infty}$ (Lions, 1996)
  $\nabla^2 u \in L_{t,x}^{\frac{4}{3},\frac{4}{3}+\varepsilon}$ (Vasseur—Y., 2021)
- Higher derivative estimate: $\nabla^\alpha u \in L_{t,x}^{\frac{4}{1+\alpha},\infty}$ (Choi—Vasseur, 2014)

Known results for uniqueness/nonuniqueness

- Weak solution in a space interpolating $L_t^2 L_x^\infty$ and $L_t^\infty L_x^3$, i.e.

$$u \in L_t^{\frac{2}{\alpha}} L_x^{\frac{3}{1-\alpha}} \text{ for some } 0 \le \alpha \le 1,$$

  are regular and unique (Ladyženskaya—Prodi—Serrin, 1960's
  Escauriaza—Seregin—Šverák, 2003)
- Mild solutions are non-unique (Buckmaster—Vicol, 2019)
  - Convex integration method for Euler (Bardos, De Lellis, Isett, Széklyhidi, Titi, Wiedemann)
- Suitable solutions are non-unique (Albritton—Brué—Colombo, 2021)
  - Instability construction for Euler (Vishik, 2018)

Maximal function associated with skewed cylinders generated by incompressible flows

Method of blow-up along trajectories

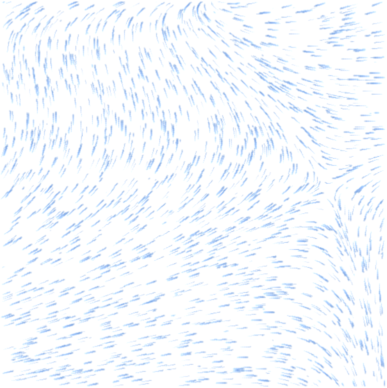
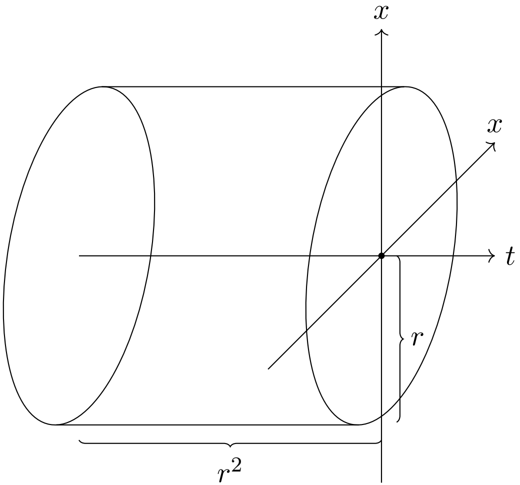


Figure: An animation of a window moving along a flow

Parabolic cylinders

$$Q_r = \{(t,x) : t \in (-r^2, 0), x \in B_r(0) \subset \mathbb{R}^3\}.$$



Parabolic cylinders along mollified flows

- Mollified flow: fix a spatial mollifier $\varphi \in C_c^\infty(B_1)$, $\int \varphi = 1$, $\varphi_\varepsilon(x) = \varepsilon^{-3}\varphi\big(\varepsilon^{-1}x\big)$, $\tilde{u}_\varepsilon = u *_x \varphi_\varepsilon$, and let $X_\varepsilon(t,x;\cdot)$ solve

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}s} X_\varepsilon(t,x;s) = \tilde{u}_\varepsilon(t, X_\varepsilon(t,x;s)), \\ &X_\varepsilon(t,x;t) = x. \end{aligned}$$

- Parabolic cylinders along X_ε are "Lagrangian cylinders in Euclidean coordinates": given (t,x), define

$$\begin{aligned} \tilde{Q}_\varepsilon(t,x) &= \big\{(t+\tau, X_\varepsilon(t,x;t+\tau)+z) \in \mathbb{R}\times\mathbb{R}^3 : (\tau,z) \in Q_\varepsilon\big\} \\ &= \big\{(s,y) \in \mathbb{R}\times\mathbb{R}^3 : t-\varepsilon^2 < s < t, |y - X_\varepsilon(t,x;t)| < \varepsilon\big\}. \end{aligned}$$

Skewed parabolic cylinders along trajectories

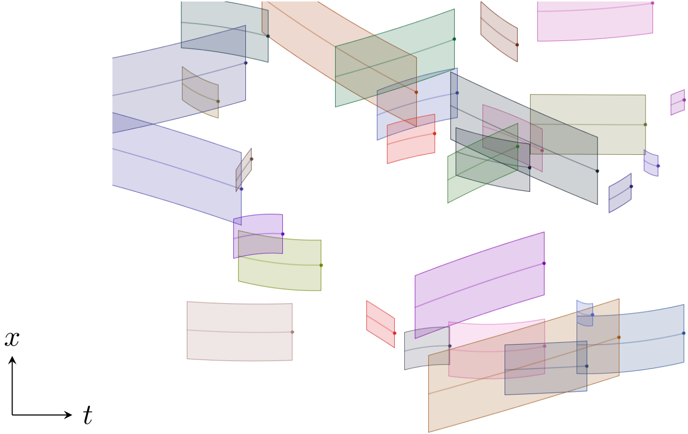


Figure: A family of skewed parabolic cylinders $\tilde{Q}_\varepsilon(t, x)$.

Admissible cylinder

$$\tilde{Q}_\varepsilon(t,x) = \big\{(s,y) \in \mathbb{R} \times \mathbb{R}^3 : t - \varepsilon^2 < s < t, |y - X_\varepsilon(t,x;t)| < \varepsilon\big\}.$$

- Assume u is divergence free, and $\mathcal{M}(\nabla u) \in L^q$ for some $1 \le q \le \infty$. Here $\mathcal{M}$ is the spatial maximal function.
- Fix a small universal $\eta_0 > 0$, $\tilde{Q}_\varepsilon(t,x)$ is an admissible cylinder if

$$\fint_{\tilde{Q}_\varepsilon(t,x)} \mathcal{M}(|\nabla u|) \, \mathrm{d}y \, \mathrm{d}s \le \eta_0 \varepsilon^{-2}.$$

- Admissibility ensures that nearby flows are close.
- We will show that for a.e. $(t,x) \in (0,T) \times \mathbb{R}^3$, for ε sufficiently small (depending on (t,x)), $\tilde{Q}_\varepsilon(t,x)$ is admissible.

Covering lemma for admissible cylinders

We show a Vitali-type covering lemma for admissible cylinders.

Lemma (Y., 2020)

Let $\mathcal{A}$ be an index set and let

$$\mathcal{Q} = \{\tilde{Q}^\alpha = \tilde{Q}_{\varepsilon_\alpha}(t^\alpha, x^\alpha) : \alpha \in \mathcal{A}\}$$

be a collection of admissible cylinders, where ε_α are uniformly bounded. Then there is a pairwise disjoint sub-collection (finite or infinite)

$$\mathcal{P} = \{\tilde{Q}^{\alpha_1}, \tilde{Q}^{\alpha_2}, \dots, \tilde{Q}^{\alpha_n}, \dots\}$$

such that

$$\sum_j \left|\tilde{Q}^{\alpha_j}\right| \geq \frac{1}{C} \left|\bigcup_{\alpha \in \mathcal{A}} \tilde{Q}^\alpha\right|.$$

Maximal function associated with admissible cylinders

- Classical maximal function: for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$,

$$\mathcal{M}f(x) := \sup_{\varepsilon>0} \frac{1}{|B_\varepsilon|} \int_{B_\varepsilon(x)} |f| \, \mathrm{d}x.$$

- We construct a **new** maximal function for admissible skewed parabolic cylinders along the trajectories of u,

$$\mathcal{M}_{\mathcal{Q}} f(t,x) := \sup \left\{ \frac{1}{|\tilde{Q}_\varepsilon|} \int_{\tilde{Q}_\varepsilon(t,x)} |f| \, \mathrm{d}x \, \mathrm{d}t : \tilde{Q}_\varepsilon(t,x) \text{ is admissible} \right\}.$$

- We have bounds on $\mathcal{M}_{\mathcal{Q}}$ similar as $\mathcal{M}$: weak-type $(1,1)$, strong-type (p,p) for $p > 1$.

Maximal function associated with admissible cylinders

$$\mathcal{M}_{\mathcal{Q}} f(t,x) := \sup \left\{ \frac{1}{|\tilde{Q}_r|} \int_{\tilde{Q}_r(t,x)} |f| \, \mathrm{d}x \, \mathrm{d}t : \tilde{Q}_r(t,x) \text{ is admissible} \right\}.$$

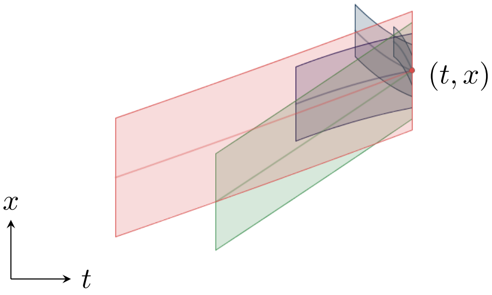


Figure: Maximal function $\mathcal{M}_{\mathcal{Q}} f(t,x)$.

Maximal function associated with admissible cylinders

Theorem (Y., 2020, to appear in Ann. Inst. Henri Poincaré (C))

There exists universal constants C_p independent of u such that

1. $\mathcal{M}_\mathcal{Q}$ *is of strong type* (∞, ∞), *i.e. for* $f \in L^\infty$,

$$\|\mathcal{M}_\mathcal{Q} f\|_{L^\infty} \le \|f\|_{L^\infty}.$$

2. $\mathcal{M}_\mathcal{Q}$ *is of weak type* $(1, 1)$, *i.e. for* $f \in L^1$, $\lambda > 0$, *the Lebesgue measure of superlevel set satisfies*

$$\mu\left(\{(t,x) : (\mathcal{M}_\mathcal{Q} f)(t,x) > \lambda\}\right) \le \frac{C_1}{\lambda}\|f\|_{L^1}.$$

3. $\mathcal{M}_\mathcal{Q}$ *is of strong type* (p, p) *for any* $1 < p < \infty$, *i.e. for* $f \in L^p$,

$$\|\mathcal{M}_\mathcal{Q} f\|_{L^p} \le C_p\|f\|_{L^p}.$$

Second derivatives estimate of the 3D incompressible Navier–Stokes equation

Main theorem

Theorem (Vasseur—Y., ARMA 2021)

Let u be a suitable weak solution in $(0,\infty)\times\mathbb{R}^3$ with initial data $u_0\in L^2$. Then for any $q>\frac{4}{3}$, $K\subset\subset(0,\infty)\times\mathbb{R}^3$, there exists a constant $C_{q,K}$ s.t.

$$\left\|\nabla^2 u\right\|_{L^{\frac{4}{3},q}(K)}\leq C_{q,K}\left(\|u_0\|_{L^2}^{\frac{3}{2}}+1\right).$$

Proof sketch

Lemma (Smallness in quadratic norm)

Let $\varphi \in C_c^\infty(B_2)$ with integral 1. There exists $\eta > 0$ s.t. if u satisfies

$$\int \varphi(x) u(t,x) \, \mathrm{d}x = 0, \qquad \textit{a.e. } t \in (-2,0),$$
$$\int_{(-2,0)\times B_2} |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}t \leq \eta,$$

then $|\Delta u| \leq 1$ in $(-1,0) \times B_1$.

Putting it back into global coordinate, it means if

$$⨍_{\tilde{Q}_\varepsilon(t,x)} |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}t \leq \eta \varepsilon^{-4},$$

then $|\Delta u(t,x)| \leq \varepsilon^{-3}$.

Proof sketch

Assume u is a suitable weak solution in $(0,T)$.

- For each $(t,x) \in (0,T) \times \mathbb{R}^3$, select $\varepsilon(t,x)$ such that either

$$\fint_{\tilde{Q}_{\varepsilon(t,x)}(t,x)} |\mathcal{M}_x(|\nabla u|)|^2 \, \mathrm{d}x \, \mathrm{d}t = \eta[\varepsilon(t,x)]^{-4},$$

  or $\varepsilon(t,x) = \sqrt{t}$ with

$$\fint_{\tilde{Q}_{\varepsilon(t,x)}(t,x)} |\mathcal{M}_x(|\nabla u|)|^2 \, \mathrm{d}x \, \mathrm{d}t < \eta[\varepsilon(t,x)]^{-4}.$$

- In either case, $|\Delta u| \le \varepsilon^{-3}$ by local theorem.
- ε^{-4} is either bounded by $\frac{1}{\eta}\mathcal{M}_{\mathcal{Q}}[\mathcal{M}_x(|\nabla u|)^2]$ or t^{-2}.
- $|\Delta u| \mathbf{1}_{\left\{|\Delta u| > t^{-\frac{3}{2}}\right\}} \in L^{\frac{4}{3},\infty}$.

Improvement of local theorem

Theorem (Smallness in almost subquadratic norm)

For $\alpha > 0$ small, there exists $\eta > 0$, $p < 2$ such that the following holds. If u has zero mean velocity in $Q_1 = [-1, 0] \times B_1$, and for some $\delta > 0$

$$\delta^{-2\alpha} \left(\int_{Q_1} |\nabla u|^p \, \mathrm{d}x \, \mathrm{d}t \right)^{\frac{2}{p}} + \delta \int_{Q_1} |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}t \le \eta,$$

then in $Q_{\frac{1}{2}} = [-\frac{1}{4}, 0] \times B_{\frac{1}{2}}$

$$|\Delta u| \le 1.$$

The novelty of this theorem is that it is purely local, depends solely on the size of ∇u, with no a priori knowledge on the pressure.

Proof of the local theorem

1. $\nabla u \in L^2_{t,x} \Rightarrow \omega \in L^\infty_t L^1_x$ (Constantin, 1990)
2. $\delta\|\omega\|_{L^\infty_t L^1_x} + (\frac{1}{\delta})^\alpha \|\nabla u\|_{L^{2-}_{t,x}} \geq \|\omega\|_{L^{2+}_t L^{2-}_x}$ (Interpolation $\alpha = 0^+$)
3. $u \in L^{2-}_t L^{6-}_x \Rightarrow u \otimes \omega \in L^{1+}_t L^{\frac{3}{2}-}_x$
4. Change of variable, let ψ and $\psi^\#$ be a pair of cut-off functions, define

$$v = -\operatorname{curl} \psi^\# \Delta^{-1}(\psi\omega),$$

   then v is called a "harmonic correction" of u, compactly supported, divergence free, $v \approx u$, $\nabla v \approx \omega$, force $\approx u \otimes \omega$, pressure $\approx u \otimes v$ (Chamorro—Lemarié-Rieusset—Mayoufi, 2018).
5. Energy inequality $\Rightarrow v \in L^\infty_t L^2_x \cap L^2_t H^1_x$.
6. De Giorgi iteration (Vasseur, 2007) $\Rightarrow v \in L^\infty$.
7. Bootstrap to higher regularity of v, $\Delta u = \Delta v$ in the interior.

Local theorem

Rescale the local theorem to the global coordinate, we have

Corollary

For $\alpha > 0$ small, there exists $\eta > 0$, $p < 2$ such that if for some $\delta > 0$,

$$\delta^{-2\alpha}\left(\fint_{\tilde{Q}_\varepsilon(t,x)} |\nabla u|^p \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{2}{p}} + \delta \fint_{\tilde{Q}_\varepsilon(t,x)} |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}t \leq \eta\varepsilon^{-4},$$

then

$$|\Delta u(t,x)| \leq \varepsilon^{-3}.$$

Recall that $\tilde{Q}_\varepsilon(t,x)$ is a **skewed parabolic cylinder** along the trajectories of $u * \varphi_\varepsilon$, centering at (t,x) with radius ε.

Main theorem

We prove the main theorem by using a similar argument as before and interpolating in Lorentz spaces.

Theorem

Let u *be a suitable weak solution in* $(0,\infty)\times\mathbb{R}^3$ *with initial data* $u_0 \in L^2$*. Then for any* $q > \frac{4}{3}$*,* $K \subset\subset (0,\infty)\times\mathbb{R}^3$*, there exists a constant* $C_{q,K}$ *s.t.*

$$\left\|\nabla^2 u\right\|_{L^{\frac{4}{3},q}(K)} \leq C_{q,K}\left(\|u_0\|_{L^2}^{\frac{3}{2}} + 1\right).$$

For regular solutions, we can bootstrap to higher regularities in vorticity, for instance $\nabla^2\omega \in L^{1,q}_{\mathrm{loc}}$ for $q > 1$.

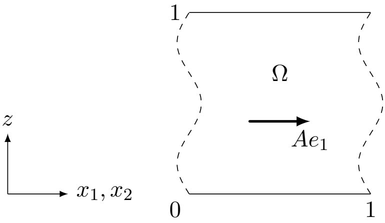
Inviscid limit problem: boundary vorticity and layer separation

3D incompressible Navier–Stokes equation

Consider the incompressible Navier–Stokes equation in a periodic tunnel $\Omega = \mathbb{T}^2 \times [0,1]$:

$$\begin{cases} \partial_t u^\nu + u^\nu \cdot \nabla u^\nu + \nabla P^\nu = \nu \Delta u^\nu & \text{in } (0,T)\times\Omega \\ \operatorname{div} u^\nu = 0 & \text{in } (0,T)\times\Omega \\ u^\nu = 0 & \text{on } (0,T)\times\partial\Omega \\ u^\nu\big|_{t=0} = u_0^\nu & \text{in } \Omega \end{cases} \qquad (\mathsf{NSE}_\nu)$$

We are interesting in the inviscid limit $\nu \to 0$ under the condition that u_0^ν converges to Ae_1 in $L^2(\Omega)$.



Asymptotic limit

- It is a major open problem to know whether the limit of u^ν converges to Ae_1.
- Only conditional results exist: the Kato criterion (1984) states that if, when $\nu \to 0$ and $u_0^\nu \to Ae_1$ in $L^2(\Omega)$:

$$\int_0^T \int_{C_\nu} \nu |\nabla u^\nu|^2 \,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t \to 0,$$

where $C_\nu = \{|z| < R\nu\} \cup \{|1-z| < R\nu\}$ is a thin region near the boundary with width of order $O(\nu)$, then

$$u^\nu \to Ae_1, \text{ in } L^\infty(0,T;L^2(\Omega)).$$

- Other conditional results: the inviscid limit holds if the solution is analytic near the boundary or if the solution possesses certain symmetry.

Turbulence and layer separation

What if the limit does not hold?

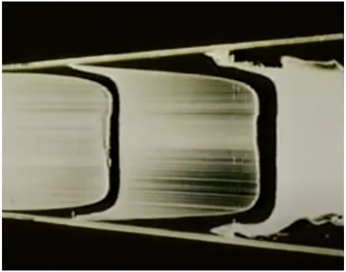


Figure: Turbulence and layer separation: the case of an airfoil and in a tunnel

Prediction of layer separation

- Formally, the asymptotic system for $\nu = 0$ is the Euler system:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = 0 & \text{in } (0,T) \times \Omega \\ \operatorname{div} u = 0 & \text{in } (0,T) \times \Omega \\ u \cdot \mathbf{n} = 0 & \text{on } (0,T) \times \partial\Omega \\ u(0,\cdot) = Ae_1 & \text{in } \Omega. \end{cases} \tag{E}$$

- The method of convex integration shows that the solution $u(t,x) = Ae_1$ of (E) is not unique (see Székelyhidi, CRAS, 2011). For every constant $C < 2$, there exists a solution with layer separation for $T < 1/A$:

$$\|u(T) - Ae_1\|_{L^2(\Omega)}^2 = CA^3T.$$

Prediction of layer separation

- Layer separation of (E):

$$\|u(T) - Ae_1\|^2_{L^2(\Omega)} = CA^3T.$$

Question:

- Is it the biggest separation possible?
- Can we get some control of the layer separation as the level of the Navier–Stokes equation?

The result

Theorem (Vasseur-Y., 2021, submitted)

For $d = 2, 3$, for every $T > 0$, for any Leray–Hopf solution u^ν to (NSE$_\nu$) *in $(0, T) \times \Omega$:*

$$\|u^\nu(T) - Ae_1\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|\nabla u^\nu\|^2_{L^2((0,T)\times\Omega)} \leq 4\|u^\nu_0 - Ae_1\|^2_{L^2(\Omega)} + CA^3T + CA^2\mathrm{Re}^{-1}\log(2 + \mathrm{Re}),$$

where $C > 0$ is a universal constant, and

$\mathrm{Re} = A/\nu$ *is the Reynolds number.*

The result

Corollary

In particular, in the inviscid limit $\nu \to 0$, *if* $u_0^\nu \to Ae_1$ *in* $L^2(\Omega)$ *and* $u^\nu \rightharpoonup u^\infty$ *in distribution up to a subsequence, then*

$$\|u^\infty(T) - Ae_1\|_{L^2(\Omega)}^2 \leq CA^3T.$$

This estimate matches the layer separation predicted by the convex integration.

Non-uniqueness and pattern predictability

- In general, non-uniqueness result by convex integration raised the question of predictability: Why can we observe patterns?
- The shear flow $u = Ae_1$ has an energy of A^2
- We prove that the layer separation has an energy of at most CA^3T at time T.
- Therefore, the perturbation stays negligible on a time span $T \ll 1/A$. This is a large time for A small (small pattern).
- It predicts the lapse of time where the pattern stays predictable.

General idea

- Maekawa and Mazzucato (The inviscid limit and boundary layers for Navier–Stokes flows, 2018):

  "Mathematically, the main difficulty in the case of the no-slip boundary condition is the lack of a priori estimates on strong enough norms to pass to the limit, which in turn is due to the lack of a useful boundary condition for vorticity or pressure."

- We show a boundary vorticity control for the Navier–Stokes equation.

Why vorticity on the boundary?

- Growth rate of the layer separation is

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^\nu - Ae_1\|_{L^2}^2 \\
&\quad = (u^\nu - Ae_1, \partial_t u^\nu) \\
&\quad = -(u^\nu - Ae_1, u^\nu \cdot \nabla u^\nu) - (u^\nu - Ae_1, \nabla P^\nu) + \nu(u^\nu - Ae_1, \Delta u^\nu) \\
&\quad = \nu(u^\nu, \Delta u^\nu) - \nu(Ae_1, \Delta u^\nu) \\
&\quad = -\nu\|\nabla u^\nu\|_{L^2}^2 - A\int_{\partial\Omega} \nu\omega_2^\nu \,\mathrm{d}x
\end{aligned}$$

- $\omega^\nu = \operatorname{curl} u^\nu$ is the vorticity of u^ν.

Boundary vorticity estimate for Navier–Stokes (intuition)

If we take the curl of (NSE$_\nu$), we have the vorticity equation,

$$\partial_t \omega + u \cdot \nabla \omega = \nu \Delta \omega + \omega \cdot \nabla u.$$

Suppose we can ignore the transport term and the boundary effect, then the regularity we could expect for ω is at best

$$\nu^2 \left\| \nabla^2 \omega \right\|_{L^1((0,T)\times\Omega)} \lesssim \nu \|\omega \cdot \nabla u\|_{L^1((0,T)\times\Omega)} \le \nu \|\nabla u^\nu\|^2_{L^2((0,T)\times\Omega)}.$$

(although parabolic regularization is false in L^1) By interpolation with $\nu\|\omega\|^2_{L^2((0,T)\times\Omega)} \le \nu\|\nabla u\|^2_{L^2((0,T)\times\Omega)}$,

$$\nu^{\frac{3}{2}} \left\| \nabla^{\frac{2}{3}} \omega \right\|^{\frac{3}{2}}_{L^{\frac{3}{2}}((0,T)\times\Omega)} \lesssim \nu \|\nabla u\|^2_{L^2((0,T)\times\Omega)}.$$

Finally the (critical) trace theorem suggests that (cheating again)

$$\|\nu\omega\|^{\frac{3}{2}}_{L^{\frac{3}{2}}((0,T)\times\partial\Omega)} \lesssim \nu \|\nabla u\|^2_{L^2((0,T)\times\Omega)}.$$

Boundary vorticity estimate for Navier–Stokes

Theorem (Boundary Regularity)

For any Leray–Hopf solution u^ν *to* (NSE$_\nu$) *in* $(0,T)\times\Omega$ *there exists a decomposition* $(0,T)\times\partial\Omega = \bigcup_i \bar{Q}^i$, *such that the following is true. Define the piecewise average on boundary* $\tilde{\omega}^\nu : (0,T)\times\partial\Omega \to \mathbb{R}$ *by*

$$\tilde{\omega}^\nu(t,x) = \fint_{\bar{Q}i} \omega^\nu \,\mathrm{d}x\,\mathrm{d}t, \qquad \text{for } (t,x) \in \bar{Q}i$$

Then we have

$$\left\| \nu \tilde{\omega}^\nu \mathbf{1}_{\{|\tilde{\omega}^\nu| > \max\{\frac{1}{t},\nu\}\}} \right\|_{L^{\frac{3}{2},\infty}((0,T)\times\partial\Omega)}^{\frac{3}{2}} \lesssim \nu \|\nabla u^\nu\|_{L^2((0,T)\times\Omega)}^2.$$

From Boundary Vorticity to Layer Separation

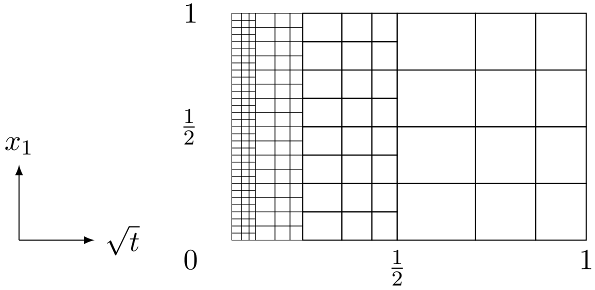
- Recall that the boundary separation is related to the size of mean vorticity

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u^\nu - Ae_1\|_{L^2(\Omega)}^2 = -\nu\|\nabla u^\nu\|_{L^2(\Omega)}^2 - A\int_{\partial\Omega} \nu\omega_2^\nu \,\mathrm{d}x,$$

- Integrate in time

$$\begin{aligned} &A\int_{\partial\Omega\times(0,T)} \nu\tilde{\omega}_2^\nu \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad \le CA^3|\partial\Omega\times(0,T)| + \frac{1}{C}\int_{\partial\Omega\times(0,T)} |\nu\tilde{\omega}^\nu|^{\frac{3}{2}} \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad \le CA^3T + \frac{1}{2}\nu\|\nabla u\|_{L^2}^2. \end{aligned}$$

The parabolic partition



A parabolic cube Q of size $4^{-k} \times (2^{-k})^d$ is said to be suitable if it touches the boundary $\partial\Omega$ and and satisfies

$$\fint_{2Q} |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}t \leq c_0(2^{-k})^{-4} \tag{S}$$

for some c_0. For each cube in the above grid that is not suitable, we dyadically dissect it into smaller cubes till suitable.

Local theorem

Theorem (Local theorem)

If Q is a suitable cube of radius 2^{-k},

$$\fint_{2Q} |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}t \leq c_0 (2^{-k})^{-4}$$

then the average boundary vorticity on $\bar{Q} = Q \cap \{z = 0\}$ is

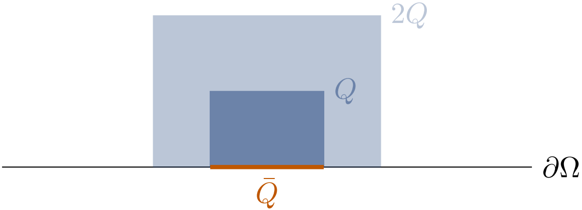
$$\tilde{\omega} = \fint_{\bar{Q}} \omega \, \mathrm{d}x' \, \mathrm{d}t \leq c_1 (2^{-k})^{-2}$$

with c_1 depending on c_0.

This lemma links the interior gradient and the mean boundary vorticity at a local level.

Local theorem

$$\fint_{2Q} |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}t \le c_0 (2^{-k})^{-4} \qquad \Longrightarrow \qquad \fint_{\bar{Q}} \omega \,\mathrm{d}x'\,\mathrm{d}t \le c_1 (2^{-k})^{-2}$$



Thank you for your attention!